The Socially Optimal Distribution of Free-Access and Private Property

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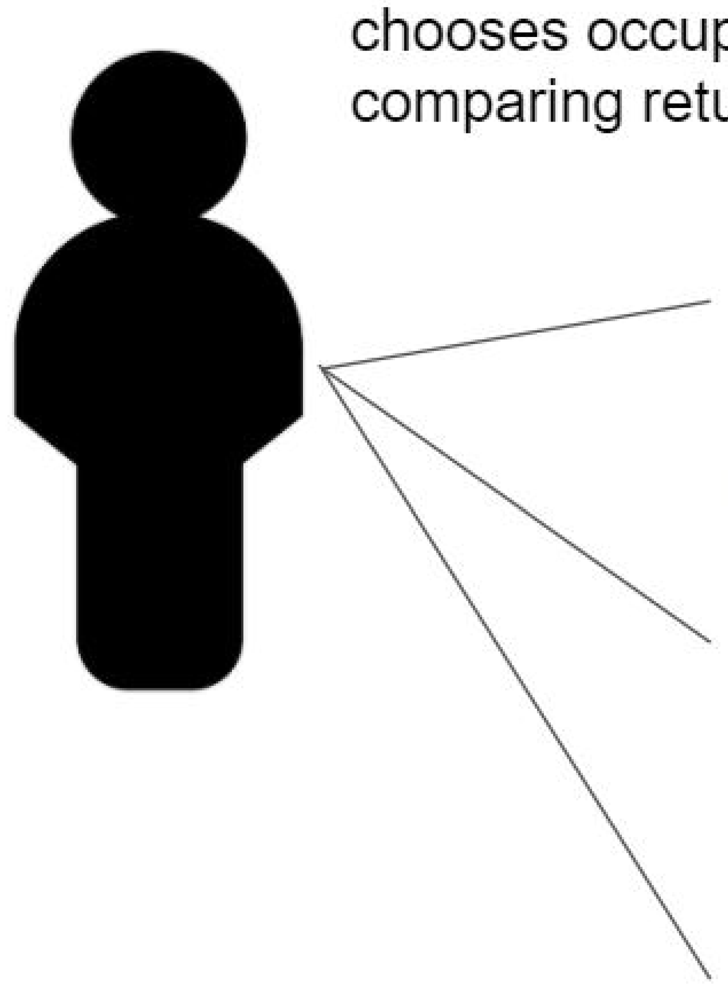
Introduction

There are a few fairly standard mathematical ways of understanding how a fixed scarce resource would be managed in private property and free-access property institutions. But, these analyses largely compare private and free-access as non-competing frameworks. Our research focuses on analyzing contexts where total privatization is not feasible or suitable resulting in a mixed framework equilibrium.

The Model and Assumptions

Utility Function $U(I, \alpha)$ chooses occupation by comparing returns from



1)Managing and Appropriating Commons as Private Property with return function:

$$\Pi(k_p, l_p, \theta, w) = G(k_p, l_p, \theta) - wl_p - C(k_p)$$

2)Laboring on remaining Free-Access Property with return function:

$$r = \frac{F(1-K_p, 1-x-L_p)}{1-x-L_p}$$

3) Earning Riskless Wage: w

Entrepreneurs or managers use the production function $\mathrm{G}(\mathrm{k_p}, \mathrm{l_p}, \theta)$ where $\mathrm{k_p}$, is the amount of land "appropriated", $\mathrm{l_p}$ is the labor hired at the prevailing competitive wage w, and θ is a random variable with density $\mathrm{g}(\theta)$. θ encodes the risk inherent in the private venture. We assume that this value is revealed after the decision is made. Entrepreneurs choose $\mathrm{k_p}$ and $\mathrm{l_p}$ to maximize profits given by

$$\Pi(\mathrm{k_p}, \mathrm{l_p}, \theta, \mathrm{w}) = \mathrm{G}(\mathrm{k_p}, \mathrm{l_p}, \theta) - \mathrm{wl_p} - \mathrm{C}(\mathrm{k_p})$$

We will assume that $\mathrm{C}(1) = \infty$, i.e. it is prohibitively expensive to enclose the entire commons.

Wage workers or "laborers" work for the entrepreneurs and earn the riskless wage w.

Let x be the proportion of entrepreneurs and $\mathrm{L_p}$ that of the wage workers. $\mathrm{K_p}$ will be the total amount of appropriated resource. The returns for sinking your labour in free access is

$$\mathrm{r} = \frac{\mathrm{F}(1-\mathrm{K_p}, 1-\mathrm{x}-\mathrm{L_p})}{1-\mathrm{x}-\mathrm{L_p}}.$$

The Existence of an Equilibrium

How do we know that this model results in an equilibrium? An equilibrium, if it exists, is characterized by $(\overline{\mathrm{x}}, \overline{\mathrm{w}}, \overline{\mathrm{r}}, \overline{\mathrm{K}}_\mathrm{p}, \overline{\mathrm{L}}_\mathrm{p})$ satisfying

- $\mathrm{U}(\overline{\mathrm{r}}) = \mathbb{E}_\theta\left[\mathrm{U}(\pi(\theta, \overline{\mathrm{w}}))\right] = \mathrm{U}(\overline{\mathrm{w}})$
- $\overline{\mathrm{r}} = \dfrac{\mathrm{F}(1-\overline{\mathrm{K}}_\mathrm{p}, 1-\overline{\mathrm{x}}-\overline{\mathrm{L}}_\mathrm{p})}{1-\overline{\mathrm{x}}-\overline{\mathrm{L}}_\mathrm{p}}$
- $\overline{\mathrm{L}}_\mathrm{p} = \overline{\mathrm{x}}\displaystyle\int \mathrm{l_p}(\theta, \overline{\mathrm{w}})\mathrm{g}(\theta)\,\mathrm{d}\,\theta; \quad \overline{\mathrm{K}}_\mathrm{p} = \overline{\mathrm{x}}\int \mathrm{k_p}(\theta, \overline{\mathrm{w}})\mathrm{g}(\theta)\,\mathrm{d}\,\theta$ with $1-\overline{\mathrm{x}}-\overline{\mathrm{L}}_\mathrm{p} \geq 0$

Let us examine the existence of an equilibrium. First, $\mathrm{U}(\mathrm{w})$ is increasing in w, while $\mathbb{E}_\theta\left[\mathrm{U}(\pi(\theta, \mathrm{w}))\right]$ is non-increasing and is strictly decreasing if, for each w, $\mathrm{l_p}(\theta, \mathrm{w})$ does not vanish identically on the support of $\mathrm{g}(\theta)$. This implies that there is a $\overline{\mathrm{w}}$ satisfying $\mathbb{E}_\theta\left[\mathrm{U}(\pi(\theta, \overline{\mathrm{w}}))\right] = \mathrm{U}(\overline{\mathrm{w}})$.

We then need to solve the following equation for x in the interval $[0, 1/\mathrm{t}(\overline{\mathrm{w}})]$.

$$\frac{\mathrm{F}(1-\mathrm{xs}(\overline{\mathrm{w}}), 1-\mathrm{xt}(\overline{\mathrm{w}}))}{1-\mathrm{xt}(\overline{\mathrm{w}})} = \overline{\mathrm{w}}$$

By working through this formula, and with minimal assumptions, it is possible to show the existence of $\overline{\mathrm{x}}$ such that $\mathrm{H}(\overline{\mathrm{x}}) = \overline{\mathrm{w}}$ follows from the continuity of H.

With these two facts, we show that an interior equilibrium will exist.

Social Welfare Functions

Now that we know there is an equilibrium, how should we judge this equilibrium? Social Welfare allows us to choose a moral framework to judge different economic outcomes.

- The simplest rule would be the utilitarian social welfare function, so $\mathrm{E^u(W)}$ where the function is increasing and $\frac{\mathrm{dE^u}}{\mathrm{dW}} > 0$ and $\frac{\mathrm{d^2E^u}}{\mathrm{dW^2}} = 0$.
- The next useful rule we could use is the continuous-prioritarian rule $\mathrm{E^p}$, where $\frac{\mathrm{dE^p}}{\mathrm{dW}} > 0$ and $\frac{\mathrm{d^2E^p}}{\mathrm{dW^2}} < 0$. This means that we are considering the benefits to those worse off to matter more.
- Finally, sufficientist social welfare functions work by adopting a threshold level $\mathrm{W}^{\text{thresh}}$ and then using a two step process. First, we compare the well-being values below the threshold. The values corresponding to a continuous-prioritarian social welfare function are ranked higher at this stage. Then, if they are tied, they are ranked by applying the utilitarian social welfare function to their corresponding threshold values above.

Evaluating Equilibrium According to the Social Welfare Functions

The social planner needs to maximize the following function for a specific rule E in order to maximize social welfare:

$$\begin{aligned}\mathrm{E(W)} &= \mathrm{E}(\mathrm{W}(\mathrm{x}^*, \mathrm{w}^*, \mathrm{r}^*, \mathrm{K}_\mathrm{p}^*, \mathrm{L}_\mathrm{p}^*)) \\ &= \mathrm{E}(\mathbb{E}_\theta\left[\mathrm{U}(\pi(\theta, \mathrm{w}^*)\right]\mathrm{x}^* + \mathrm{U}(\mathrm{w}^*)\mathrm{L}_\mathrm{p}^* + \mathrm{U}(\mathrm{r}^*)(1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*)\end{aligned}$$

Thus, to find a local maximum, $\nabla\mathrm{E} = \vec{0}$ where it satisfies a second derivative test. Notice that

$$\frac{\partial \mathrm{W}}{\partial \mathrm{K}_\mathrm{p}^*} = -\mathrm{F}_{\mathrm{K}_\mathrm{p}^*}(1-\mathrm{K}_\mathrm{p}^*, 1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*)\mathrm{U}'(\frac{\mathrm{F}(1-\mathrm{K}_\mathrm{p}^*, 1-\mathrm{x}^*-\mathrm{L_p})}{1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*})$$

Since $\mathrm{U}'(\frac{\mathrm{F}(1-\mathrm{K}_\mathrm{p}^*, 1-\mathrm{x}^*-\mathrm{L_p})}{1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*}) > 0$ because $1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^* > 0$ and $\mathrm{Kp} \neq 1$ assuming an interior solution (equilibrium point), then for $\frac{\partial \mathrm{W}}{\partial \mathrm{K}_\mathrm{p}^*} = 0$ we need $\mathrm{F}_{\mathrm{K}_\mathrm{p}^*}(1-\mathrm{K}_\mathrm{p}^*, 1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*) = 0$. Since that $\mathrm{F}_{\mathrm{K}_\mathrm{p}^*} \neq 0$ for any amount of capital on the interval, then we know that $\frac{\partial \mathrm{W}}{\partial \mathrm{K}_\mathrm{p}^*} \neq 0$ for the relevant intervals. Thus, for the equilibrium point $\mathrm{E}'(\mathrm{W}) = 0$ must hold for to maximise social welfare. At equilibrium

$$\mathrm{E}'(\mathrm{W}) = \mathrm{E}'(\mathbb{E}_\theta\left[\mathrm{U}(\pi(\theta, \mathrm{w}^*)\right]\mathrm{x}^* + \mathrm{U}(\mathrm{w}^*)\mathrm{L}_\mathrm{p}^* + \mathrm{U}(\frac{\mathrm{F}(1-\mathrm{K}_\mathrm{p}^*, 1-\mathrm{x}^*-\mathrm{L_p})}{1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*})(1-\mathrm{x}^*-\mathrm{L}_\mathrm{p}^*))$$

$$\mathrm{E}'(\mathrm{W}) = \mathrm{E}'(\mathrm{U}(\overline{\mathrm{w}}))$$

Since $\mathrm{U}'(\overline{\mathrm{w}}) \neq 0$ then the equilibrium is not optimal for utilitarian or continuous-prioritarian outright due to positive first derivatives. For whatever threshold chosen, since we know it is not optimal for the general continuous-prioritarian, we can know that the equilibrium is not optimal.

Further Questions

We can conclusively say that by any rule chosen for social welfare analysis, the equilibrium reached by the market by individual choices is not socially optimal **for a homogeneous population** when it comes to risk aversion and ability. What this does not consider is the cost of having some non-market system to reach social optimally. Additionally, we don't know how far from social optimally the free market equilibrium is because of all the non-specified functions. Thus, more analysis would be needed to determine how to reach social optimality or if the cost of adjustment would be larger than the difference to optimality and the market equilibrium. Additionally, when skill or risk aversion is non-homogeneous, a more realistic scenario, this model contains the framework to vary these variables but any results would require further inquiry.