Properties of the preshears of rosette mappings

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Abstract

We explore properties of the preshears of rosette harmonic mappings. These mappings were discovered by generalizing the harmonic mapping in the form of a 6-noded rosette that was obtained as the conformal mapping onto a symmetric cross. We consider the preshears for each integer n with $n \geq 3$.By construction, the preshear conformal mappings have exterior angles at vertices that are $\pm\pi/2$ or π, and the image regions consist of $n-3$ block-shaped regions.

Introduction and Description of the Problem

The rosette harmonic mappings were discovered by an application of the shear construction to a conformal mapping with symmetry group D_4. While the shear construction does not normally produce a symmetric or even bounded image, the shear construction in this case to a harmonic mapping of the disk onto a symmetrically shaped 6-noded "rosette" with symmetry group D_6. A similar shear of the mapping onto the same object, but rotated through a one eighth rotation of the circle, also yielded a harmonic mapping of the disk onto a closed 6-cusped rosette region with symmetry group D_6 [**?**]. The formulae obtained generalized easily to yield a larger family of harmonic mappings with n-cusps, or n-nodes and whose images also have D_n symmetry, where n is an integer with $n \geq 3$. Moreover, for each such n, a less symmetric family of counterparts f_β were found for every $\beta \in \mathbb{R}$, and provided β is not a multiple of $\pi/2$, the image of the unit disk under f_β has symmetry group C_n rather than D_n. The fact that the resultant mappings are harmonic mappings is proved in [1]. Also described there are the dimensions and locations of features of the images of the unit disk under the mappings f_β.

Goal: To investigate the properties of the preshears of rosette mappings.

Many conformal mappings result in "noise", when sheared. Because rosettes are so symmetrical and predictably structured, the preshears of such mappings are of great interest.

The Rosette Mapping

Let U denote the unit disk. Let n be an integer, $n \geq 2$ and $\beta \in \mathbb{R}$. For $z \in \bar{U}$, consider the hypergeometric functions $H_n(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2n}, 1+\frac{1}{2n}, z\right)$ and $G_n(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-\frac{1}{2n}, \frac{3}{2}-\frac{1}{2n}, z\right)$. We also define functions

$$h_n(z) = z\, H_n\left(z^{2n}\right), \quad g_n(z) = \frac{z^{n-1}}{n-1} G_n\left(z^{2n}\right), \ z \in \bar{U},$$

rotations of which form the analytic and co-analytic parts of the harmonic mapping

$$f_\beta(z) = e^{i\beta/2} h_n(z) + e^{-i\beta/2}\overline{g_n(z)}, \ z \in \bar{U}.$$

We refer to $f_\beta(z)$ as a rosette mapping.

Theorem 1. *Let n be an integer where $n \geq 3$, and let $\beta \in \mathbb{R}$.*

Let f_β be a rosette mapping. Let F_β be the conformal mapping $F_\beta = e^{\frac{i\beta}{2}} h_n - e^{\frac{i\beta}{2}} g_n$. Then f_β is the shear of F_β, with dilatation $\omega = z^{n-2}$.

Definition 1. We define the Schwartz-Christoffel formula to be

$$\phi(z) = A_1 \int_0^z \frac{1}{(w-\zeta_1)^{b_1}(w-\zeta_2)^{b_2}\cdots(w-\zeta_n)^{b_n}} dz + A_2,$$

where A_1, A_2 are constants.

The Schwartz-Christoffel formula is used to construct the preshears of our rosette mappings. We see that indeed, our preshears as defined may be expressed in a form indicative of the influence of Schwartz-Christoffel.

Corollary 1. *Let $\beta \in (-\pi/2, \pi/2]$. Then F_β is a rotation of F_0 by an angle of $\beta/2$, that is*

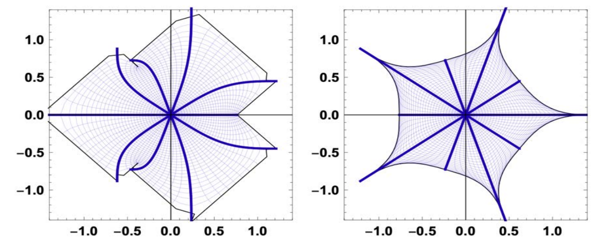
$$F_\beta = e^{i\beta/2} F_0.$$

Theorem 2. *Only the b values $\frac{1}{2}, -\frac{1}{2}$, and -1 appear in the Schwarz-Christoffel formula for preshears of rosette mappings.*

Definition 2. A **divot** is a vertex with exterior angle $-\pi/2$, corresponding to $b = -\frac{1}{2}$ on the Schwarz-Christoffel representation of the preshear. A **crevice** is a vertex with exterior angle $-\pi$. A **bead** is is a vertex with exterior angle $+\pi/2$.

Image of a Rosette Mapping on the complex plane

Displayed below is a rosette mapping for $n = 5$, along with its preshear.



References

[1] J. McDougall and L. Stierman, "Rosette harmonic mappings," 2020.

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