Resonant Tunneling in Quantum Mechanics

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Abstract

In Quantum Mechanics, often the goal of solving some problem is to determine the probability that a particle is in some specified location at some given time. Our group's quantum problem of interest is the resonant tunneling phenomena. In this setup, it is most useful to find the probability that a particle tunnels through the barriers- the transmission amplitude. There are very few tunneling problems one may solve analytically in Quantum Mechanics. In our research, we began by solving many of the more simple problems analytically to build a basis of understanding for how we would solve the complex resonance cases. Our method was a traditional Quantum Mechanical approach of assigning wave functions to specific regions of potential energy using the Schrodinger Equation. Next, using systems of equations, complex analysis, the equation for transmission amplitude, and dimensional analysis, we expressed our solution for the transmission amplitude. We later approached more variable setups of resonant tunneling using the popular WKB approximation. Our results supported our hypothesis of spikes in transmission probability for specific sweet-spot energy levels..

Introduction

The time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

Throughout our research we worked with various different potential barrier setups and attempted to derive the reflection and transmission coefficients for each. Although our overall methodology of analytically solving for solutions or going about approximating solutions was generally the same for each setup, we spent much more time on some problems than others. We progressed from solving simple potential function frameworks to approximating more complicated ones throughout the summer. One particular setup we put a lot of focus into was the resonant tunneling potential function that featured a particle tunneling through two square barriers in succession.

Materials

The main program used for this project was Mathematica to do the more complicated calculations.

We did most of our background learning using *Mathematical Methods for Physicists: A Comprehensive Guide* by Arfken, Weber and Harris, as well as using *Introduction to Quantum Mechanics* by Griffiths.

Latex was used to format the note papers.

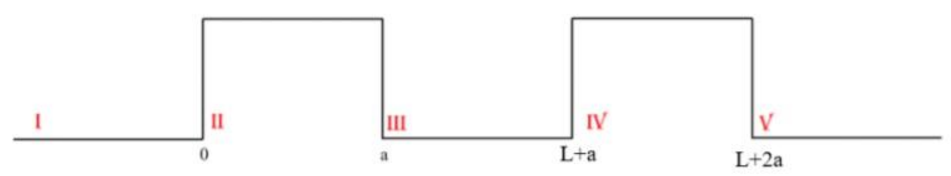
Methodology

Rather than this research requiring a laboratory to conduct it, the majority of our summer was spent solving the Schrodinger equations for various potential barrier scenarios.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \frac{d^2\psi}{dx^2} = -k^2\psi \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

Using this method, we were able to write systems of equations for each barrier set up. For the double square barrier,



At :

$$A_1 + B_1 = C_1 + D_1 \qquad (89)$$

$$A_1 - B_1 = \frac{\kappa}{ik}(C_1 - D_1) \qquad (90)$$

At x = a;

$$C_1e^{\kappa a} + D_1e^{-\kappa a} = A_2e^{ika} + B_2e^{-ika} \qquad (91)$$

$$\kappa C_1e^{\kappa a} - \kappa D_1e^{-\kappa a} = ikA_2e^{ika} - ikB_2e^{-ika} \qquad (92)$$

At x = L+a;

$$A_2e^{ik(L+a)} + B_2e^{-ik(L+a)} = C_2e^{\kappa(L+a)} + D_2e^{-\kappa(L+a)} \qquad (93)$$

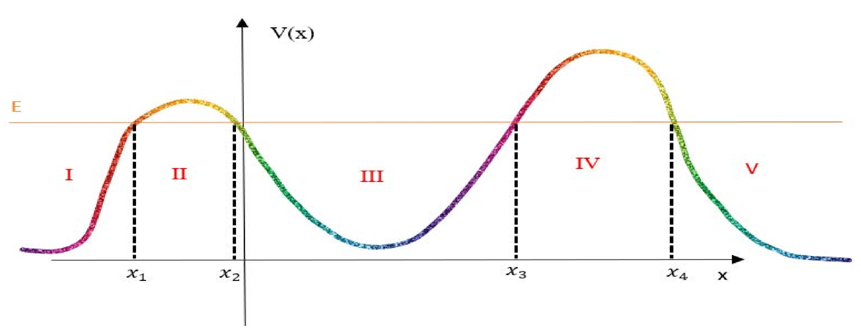
$$ikA_2e^{i(L+a)} - ikB_e^{-ik(L+a)} = \kappa C_2e^{\kappa(L+a)} - \kappa D_2e^{-\kappa(L+a)} \qquad (94)$$

At x = L+2a;

$$C_2e^{\kappa(2a+L)} + D_2e^{-\kappa(2a+L)} = A_3e^{ik(2a+L)} \qquad (95)$$

$$\kappa C_2e^{\kappa(2a+L)} - \kappa D_2e^{-\kappa(2a+L)} = ikA_3e^{ik(2a+L)} \qquad (96)$$

Using Mathematica, we were able to solve for the probability of a particle ending up on the right of the second barrier. Using the results from this, we were able to set a up a similar situation. However, since the regions II and IV are non-classical, we had to use the connection formulas seen below.



$$\frac{2A}{\sqrt{p(x)}}\cos\left(\frac{1}{\hbar}\int_x^a p(x)\,dx - \frac{\pi}{4}\right) - \frac{B}{\sqrt{p(x)}}\sin\left(\frac{1}{\hbar}\int_x^a p(x)\,dx - \frac{\pi}{4}\right) \leftrightarrow$$
$$\frac{A}{\sqrt{|p(x)|}}\exp\left(-\frac{1}{\hbar}\int_a^x |p(x)|\,dx\right) + \frac{B}{\sqrt{|p(x)|}}\exp\left(\frac{1}{\hbar}\int_a^x |p(x)|\,dx\right)$$

$$\frac{A}{\sqrt{|p(x)|}}\exp\left(-\frac{1}{\hbar}\int_x^b |p(x)|\,dx\right) + \frac{B}{\sqrt{|p(x)|}}\exp\left(\frac{1}{\hbar}\int_x^b |p(x)|\,dx\right) \leftrightarrow$$
$$\frac{2A}{\sqrt{p(x)}}\cos\left(\frac{1}{\hbar}\int_b^x p(x)\,dx - \frac{\pi}{4}\right) - \frac{B}{\sqrt{p(x)}}\sin\left(\frac{1}{\hbar}\int_b^x p(x)\,dx - \frac{\pi}{4}\right)$$

Results

The results we will be presenting are for the double sqaure barrier resonant tunneling setup. After non-dimensionalizing, our final result to graph to Mathematica ended up being

$$(16(\epsilon-1)^2\epsilon^2)/$$
$$(\sinh[\alpha\sqrt{2-2\epsilon}]^2(4(1-\epsilon)\sqrt{-(-1+\epsilon)\epsilon}\cosh[\alpha\sqrt{2-2\epsilon}] + \sin[2\sqrt{2}\beta\sqrt{\epsilon}]\sinh[\alpha\sqrt{2-2\epsilon}]^2)^2$$
$$+(-4(-1+\epsilon)\epsilon\cosh[\alpha\sqrt{2-2\epsilon}]^2 + (-(1-2\epsilon)^2 + \cos[2\sqrt{2}\beta\sqrt{\epsilon}])\sinh[\alpha\sqrt{2-2\epsilon}]^2)^2)$$

We ended up plotting probability as a function to the ratio of initial Energy to Potential Energy and keeping the alpha and beta terms as variables we can manipulate with a slider. Our results were mathematically beautiful! We found that square barrier resonant tunneling highly favors some energies over others when it comes to tunneling

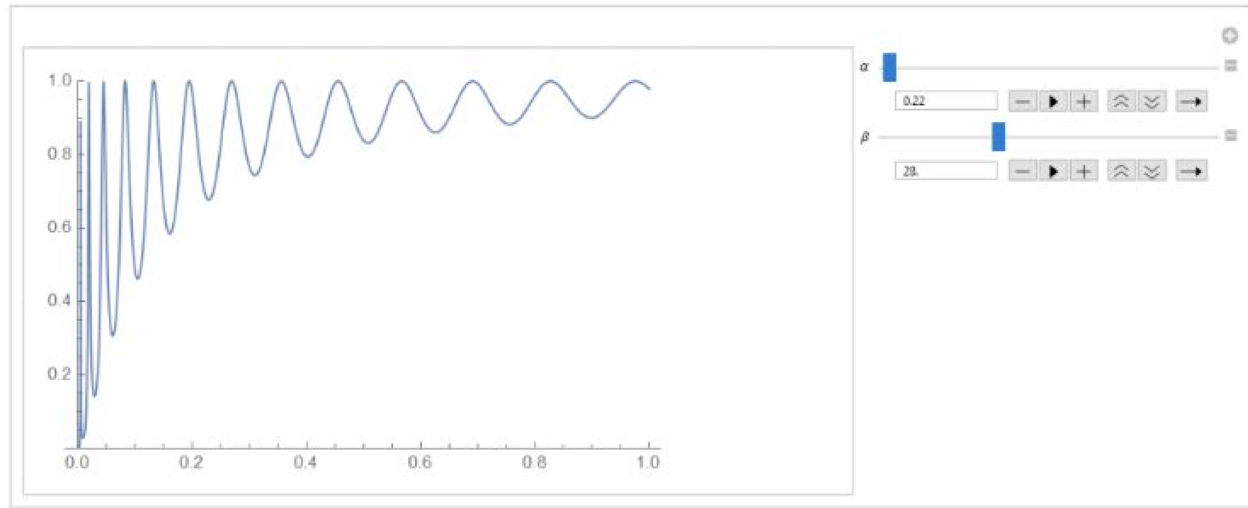


Figure 4: Probability Function 1

The second graph shows the very specific Energy level ratios that have nearly a one-hundred percent chance of tunneling for this given set up. The width of the barriers, distance between them are the two initial conditions that we are able to adjust in order to see how the energy to tunneling probability changes for each energy to potential energy ratio.

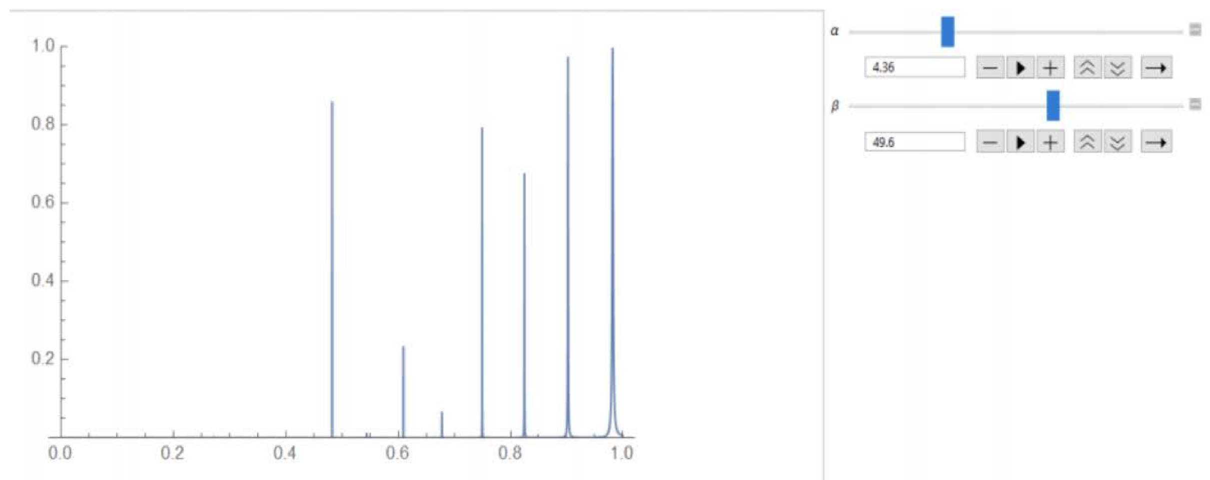


Figure 5: Probability Function 2

After completing the double square barrier problem we used the skills we had acquired along with a technique called the Wentzel-Kramers-Brillioun approximation to begin working on a much more general double barrier potential function set up. We ended up with this expression:

$$64e^{\frac{2(k+p)}{\hbar}}/$$
$$-\left(16e^{\frac{4k}{\hbar}}-1\right)\left(4e^{\frac{2p}{\hbar}}+1\right)\left(4e^{\frac{2p}{\hbar}}\sin\left(\frac{2m}{\hbar}\right)+\cos\left(\frac{2m}{\hbar}\right)\right)$$
$$+4\left(1-4e^{\frac{2k}{\hbar}}\right)^2 e^{\frac{2p}{\hbar}} + 16\left(16e^{\frac{4k}{\hbar}}+1\right)e^{\frac{4p}{\hbar}} + 16e^{\frac{4k}{\hbar}} + 1$$